Dichotomy between deterministic and probabilistic models in countably additive effectus theory

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Effectus theory is a relatively new approach to categorical logic that can be seen as an abstract form of generalized probabilistic theories (GPTs). While the scalars of a GPT are always the real unit interval $[0,1]$, in an effectus they can form any *effect monoid*. Hence, there are quite exotic effectuses resulting from more pathological effect monoids.

In this paper we introduce σ-*effectuses*, where certain countable sums of morphisms are defined. We study in particular σ-effectuses where unnormalized states can be normalized. We show that a non-trivial σ-effectus with normalization has as scalars either the two-element effect monoid $\{0,1\}$ or the real unit interval $[0,1]$. When states and/or predicates separate the morphisms we find that in the $\{0,1\}$ case the category must embed into the category of sets and partial functions (and hence the category of Boolean algebras), showing that it implements a deterministic model, while in the $[0,1]$ case we find it embeds into the category of Banach order-unit spaces and of Banach pre-base-norm spaces (satisfying additional properties), recovering the structure present in GPTs.

Hence, from abstract categorical and operational considerations we find a dichotomy between deterministic and convex probabilistic models of physical theories.

1 Introduction

In the widely used generalized probabilistic theories (GPTs), see e.g. [2–4, 6], measurement and probability are of central importance. A system in a GPT is described by a real vector space corresponding to the states of the system, while the *effects*, two-outcome measurements, lie in the dual vector space.

Effectus theory, introduced by Jacobs [25], is an approach to categorical logic that can describe deterministic, probabilistic or quantum logic; see also [10, 11, 38]. An effectus is analogous to a GPT where the real interval $[0,1]$ of probabilities is replaced by an *effect monoid* M. As a result, states form an *(abstract) convex set* over M instead of lying in a real vector space, while effects form an *effect module* over M. Tull [35, 36] showed that effectuses can be understood as certain *operational theories* in the style of Chiribella et al. [8, 12].

Taking the effect monoid of scalars in an effectus to be $[0,1]$, the effectus is quite close in structure to that of a GPT (especially when operationally motivated state/effect separation properties are imposed, cf. Section 4). Instead taking the scalars to be the Booleans $\{0,1\}$, the effectus describes a deterministic theory where every predicate either holds with certainty on each state, or does not hold at all. Every effect monoid can form the set of scalars of an effectus (Propositions 25 and 30), and since there exist quite pathological effect monoids, there are exotic effectuses that have no easy comparison to GPTs or deterministic theories.

In this paper we show that this situation changes when we consider effectuses with some additional structure. A central notion in effectus theory is the existence of certain sums of morphisms. In this paper we introduce σ-*effectuses*, where we strengthen this to the existence of certain countable sums of morphisms, based on the well-established notion of *partially additive categories* [1, 32]. The extension allows

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measurements with countably many outcomes (see Remark 10), and also it generalizes the assumption that one can form countable mixture of states (e.g. [14, 15, 31]). In a σ-effectus the scalars form an *ω-complete* effect monoid (i.e. where suprema of increasing sequences exist). In [37] these were shown to always embed in a direct sum of a Boolean algebra and the unit interval of a commutative C*-algebra. This characterization shows that the scalars in a σ-effectus are necessarily well-behaved. This has several immediate consequences for σ-effectuses, such as that the scalars are always commutative.

A natural condition, which in GPTs is usually assumed implicitly, is that every unnormalized state can be *normalized*. We present a number of equivalent conditions for a σ-effectus to allow normalization, one of which is that the scalars must be one of $\{0\}$, $\{0,1\}$ and $[0,1]$.

Hence σ-effectuses with normalization come in three different types. When the scalars of an effectus are $\{0\}$, the category is equivalent to the trivial single object category, and hence this type is not particularly interesting. If instead the scalars are $\{0,1\}$, the σ-effectus describes a deterministic theory where each predicate (does not) hold with certainty. If we additionally assume that states separate morphisms, every such σ-effectus $\mathbf{C}$ has a faithful morphism of σ-effectuses into the category **Pfn** of sets and partial functions (and hence the category of Boolean algebras). And finally, if the scalars are $[0,1]$ we have a GPT-like convex probabilistic theory. Under suitable separation assumptions, the σ-effectus faithfully embeds into a category of *order-unit spaces* and of *(pre-)base-norm spaces*, which are ordered vector spaces used in GPTs [3, 6]. Our results then establish, from purely categorical and order-theoretic considerations, a dichotomy between classical deterministic and convex probabilistic models.

2 Preliminaries

We recall the well-established notions of partially σ-additive monoids and partially σ-additive categories[1] due to Arbib and Manes [1, 32] and their finitary counterparts. Further details can be found in [10].

Definition 1. A **partial commutative monoid** (**PCM**) is a set X with an element $0 \in X$ and a *partial* binary operation $\ovee \colon X \times X \rightharpoonup X$ such that $(x \ovee y) \ovee z = x \ovee (y \ovee z)$, $x \ovee y = y \ovee x$, and $0 \ovee x = x$ for all $x, y, z \in X$, where '$=$' is taken to be a *Kleene equality*: 'if either side is defined, then so is the other, and they are equal'. Hence an equation like $x \ovee y = z$ is taken to mean that $x \ovee y$ is defined in addition to the equality $x \ovee y = z$. We will write $x \perp y$ to denote $x \ovee y$ is defined.

Let M, N and L be PCMs. A function $f : M \to N$ is **additive** if $f(0) = 0$ and $f(x) \ovee f(y) = f(x \ovee y)$ for all $x \perp y$ in M. A function $g \colon M \times N \to L$ is **biadditive** if $g(x, -) \colon N \to L$ and $g(-, y) \colon M \to L$ are additive for all $x \in M$ and $y \in N$.

A finite sequence $x_1, \ldots, x_n$ in a PCM M is **summable** if $\bigovee_{i=1}^n x_i := (\cdots (x_1 \ovee x_2) \ovee \cdots) \ovee x_n$ is defined in M. The sum $\bigovee_{i=1}^n x_i$ does not depend on the ordering, yielding a partial addition operation on finite families. Arbib and Manes defined the notion of partial addition extended to countable families.

Definition 2. A **partially σ-additive monoid** (**σ-PAM**) is a nonempty set M equipped with a partial operation $\bigovee$ that sends a countable family $(x_j)_{j \in J}$ of elements in M to an element $\bigovee_{j \in J} x_j$ in M, satisfying the three axioms below. We say that $(x_j)_{j \in J}$ is **summable** if $\bigovee_{j \in J} x_j$ is defined.

- **Partition-associativity axiom**: For each countable family $(x_j)_{j \in J}$ and each countable partition $J = \biguplus_{k \in K} J_k$, the family $(x_j)_{j \in J}$ is summable if and only if $(x_j)_{j \in J_k}$ is summable for each $k \in K$ and $(\bigovee_{j \in J_k} x_j)_{k \in K}$ is summable. In that case, one has $\bigovee_{j \in J} x_j = \bigovee_{k \in K} \bigovee_{j \in J_k} x_j$.
- **Unary sum axiom**: Each singleton $\{x_j\}_{j \in \{*\}}$ is summable and satisfies $\bigovee_{j \in \{*\}} x_j = x_*$.

[1] Arbib and Manes called these notions simply 'partially additive monoids' and 'partially additive categories'. We here added 'σ' in order to emphasize their countable structures and to avoid confusion with their finitary counterparts.

- **Limit axiom**: A countable family $(x_j)_{j\in J}$ is summable whenever for any finite subset $F\subseteq J$, the subfamily $(x_j)_{j\in F}$ is summable.

Note that every σ-PAM is a PCM via $x_1 \ovee x_2 = \bigovee_{i\in\{1,2\}} x_i$ and $0=\bigovee\emptyset$.

Let M,N and L be σ-PAMs. A function $f\colon M\to N$ is **σ-additive** if for any summable family $(x_j)_{j\in J}$ in M, the family $(f(x_j))_{j\in J}$ is summable in N and $f(\bigovee_{j\in J}(x_j)) = \bigovee_{j\in J} f(x_j)$. A function $g\colon M\times N\to L$ is **σ-biadditive** if $g(x,-)\colon N\to L$ and $g(-,y)\colon M\to L$ are σ-additive for all $x\in M$ and $y\in N$.

Following Arbib and Manes, we will introduce a notion of categories equipped with partial addition of morphisms. But first we require some definitions. We say a category $\mathbf{C}$ is **enriched over PCMs** (resp. **enriched over σ-PAMs**) if each homset $\mathbf{C}(A,B)$ is a PCM (resp. σ-PAM) and each composition map $\circ\colon \mathbf{C}(B,C)\times\mathbf{C}(A,B)\to\mathbf{C}(A,C)$ is (σ-)biadditive. When $\mathbf{C}$ is a category with zero morphisms $0\colon A\to B$ (such as when it is enriched over PCMs), each coproduct $\coprod_{j\in J} A_j$ in $\mathbf{C}$ has **partial projections** $\rhd_i\colon \coprod_{j\in J} A_j\to A_i$ characterized by $\rhd_i\circ\kappa_i=\mathrm{id}$ and $\rhd_i\circ\kappa_k=0$ for $k\neq i$. Here $\kappa_i\colon A_i\to\coprod_{j\in J}A_j$ denote coprojections. A family $(f_j\colon B\to A_j)_{j\in J}$ of morphisms is **compatible** if there exists an $f\colon B\to\coprod_{j\in J}A_j$ such that $\rhd_j\circ f=f_j$ for each $j\in J$.

Definition 3. A **finitely partially additive category** (resp. **partially σ-additive category**) is a category with finite (resp. countable) coproducts that is enriched over PCMs (resp. over σ-PAMs) satisfying the following two axioms relating coproducts to the additive structure.

- **Compatible sum axiom**: Compatible pairs of morphisms $f,g\colon A\to B$ (resp. countable families $(f_j\colon A\to B)_{j\in J}$) are summable in $\mathbf{C}(A,B)$.
- **Untying axiom**: If $f,g\colon A\to B$ are summable, then $\kappa_1\circ f,\kappa_2\circ g\colon A\to B+B$ are summable too.

We write 'finPAC' for 'finitely partially additive category' and 'σ-PAC' for 'partially σ-additive category'.

Remark 4. Fin/σ-PACs can be characterized in a more categorically simple manner as categories with finite/countable coproducts, zero maps, and some other axioms [10, § 3.8.1], [1, § 5].

Before moving on to effectuses, we need a final additional type of structure.

Definition 5. An **effect algebra** [16] is a PCM $(E,\ovee,0)$ with a 'top' element $1\in E$ such that for each $a\in E$, (i) there is a unique $a^\perp\in E$ (called the orthosupplement) such that $a\ovee a^\perp=1$; and (ii) $a\perp 1$ implies $a=0$. We write **EA** for the category of effect algebras and additive maps.

Note that effect algebras are posets with $a\leq b$ when $a\ovee c=b$ for some c.

Remark 6. The usual notion of morphisms $f\colon E\to D$ between effect algebras are additionally **unital** in the sense that $f(1)=1$. Our morphisms in **EA** however are only 'subunital', i.e. $f(1)\leq 1$. We make this change because we will use effectuses *in partial form* which denotes a category with 'partial' morphisms; see Remark 13 below. We will require a similar change in morphisms in several other categories.

Example 7. A Boolean algebra $(B,0,1,\vee,\perp)$ is an effect algebra with $(\ )^\perp$ the regular complement, $a\perp b$ iff $a\wedge b=0$ and in that case $a\ovee b\equiv a\vee b$.

Example 8. Let $B(H)$ be the space of bounded operators on a Hilbert space H equipped with the standard partial order. Its *effects* are the operators $A\in B(H)$ satisfying $0\leq A\leq 1$. The space of effects $[0,1]_{B(H)}$ is then an effect algebra with $A\perp B$ when $A+B\leq 1$ and then $A\ovee B\equiv A+B$.

3 Effectuses and σ-effectuses

In this section, we present the basic theory of σ-effectuses. We describe effectuses as well, showing how their theory [11,25] can be naturally extended to the σ-additive setting. In addition, we introduce a notion of *(σ-)weight modules* to axiomatize the structure of substates.

A (σ-)effectus is basically a fin/σ-PAC with a special unit object representing 'no system'. The morphisms to the unit object are then the ways in which a system can be 'destroyed' or 'measured' and hence are the effects of the system. They are assumed to form effect algebras.

Definition 9. An **effectus** (in partial form, see Remark 13 below) is a finPAC $\mathbf{C}$ with a distinguished 'unit' object $I \in \mathbf{C}$ satisfying the following conditions.

(i) For each $A \in \mathbf{C}$, the hom-PCM $\mathbf{C}(A,I)$ is an effect algebra. We write $\mathbf{1}_A$ and $\mathbf{0}_A = 0_{AI}$ for the top and bottom in $\mathbf{C}(A,I)$.

(ii) $\mathbf{1}_B \circ f = \mathbf{0}_A$ implies $f = 0_{AB}$ for all $f\colon A \to B$.

(iii) $\mathbf{1}_B \circ f \perp \mathbf{1}_B \circ g$ implies $f \perp g$ for all $f,g\colon A \to B$.

A **σ-effectus** is a σ-PAC $\mathbf{C}$ with a distinguished object $I \in \mathbf{C}$ satisfying the same conditions (i)–(iii).

A **morphism of effectuses** (resp. **σ-effectuses**) $(\mathbf{C}, I_{\mathbf{C}}) \to (\mathbf{D}, I_{\mathbf{D}})$ is a functor $F\colon \mathbf{C} \to \mathbf{D}$ that preserves finite (resp. countable) coproducts and 'preserves the unit' in the sense that there is an isomorphism $u\colon I_{\mathbf{D}} \to FI_{\mathbf{C}}$ such that $F\mathbf{1}_A = u \circ \mathbf{1}_{FA}$ for each $A \in \mathbf{C}$. I.e. the diagram on the right commutes.

$$\begin{tikzcd} FA \arrow[d, "\mathbf{1}_{FA}"'] \arrow[dr, "F\mathbf{1}_A"] & \\ I_{\mathbf{D}} \arrow[r, "\cong", "u"'] & FI_{\mathbf{C}} \end{tikzcd}$$

A morphism $f\colon A \to B$ in a (σ-)effectus is **total** if $\mathbf{1}_B \circ f = \mathbf{1}_A$. The total morphisms form a (wide) subcategory $\mathrm{Tot}(\mathbf{C}) \hookrightarrow \mathbf{C}$. A **predicate** on A is a morphism of type $p\colon A \to I$. A **state** on A is a morphism of type $\omega\colon I \to A$ that is total, i.e. satisfies $\mathbf{1}_A \circ \omega = \mathbf{1}_I$. A **substate** on A is any (not necessarily total) morphism $\omega\colon I \to A$. The morphisms $s\colon I \to I$ are called **scalars** and we view them as abstract probabilities. We write $\mathrm{Pred}(A) = \mathbf{C}(A,I)$, $\mathrm{St}(A) = \mathrm{Tot}(\mathbf{C})(I,A)$, $\mathrm{St}_{\leq}(A) = \mathbf{C}(I,A)$ for the set of predicates, states and substates respectively.

Remark 10. As studied by Tull [35, 36], one can interpret a (σ-)effectus as an *operational theory* in the style of Chiribella et al. [7, 8, 12] (see also [10, § 6.1, 6.2]). In their terminology, each morphism $f\colon A \to B$ is called an **event**. A **test** from system A to B is then a summable family of events $(f_x\colon A \to B)_{x \in X}$ such that $\bigovee_{x \in X} f_x$ is total. The indexing set $x \in X$ is understood as the set of outcomes of the test. In particular, a 'preparation' test $(\omega_x\colon I \to A)_{x \in X}$ consists of substates and an 'observation' test $(p_x\colon A \to I)_{x \in X}$ consists of predicates. Each 'closed' test $(s_x\colon I \to I)_{x \in X}$, which satisfies $\bigovee_{x \in X} s_x = 1$, describes the abstract probability s_x that the test yields an outcome $x \in X$.

Example 11. A **partial function** $f\colon X \rightharpoonup Y$ is a function of sets where for each $x \in X$, $f(x)$ is either an element of Y or undefined. We write $\mathrm{Dom}(f) \subseteq X$ for the domain of definition, i.e. the set of $x \in X$ where $f(x)$ is defined. Partial functions compose in the obvious way. The category of sets and partial functions **Pfn** is a σ-effectus with the singleton $I = \{*\}$ as unit. Partial functions are summable when they have disjoint domains of definition. Such partial functions can be merged into one partial function in the obvious way, which defines the sum. Indeed, **Pfn** is the prototypical example of σ-PAC in [1, 32]. For a set X, we have $\mathrm{St}(X) \cong X$ and $\mathrm{Pred}(X) \cong \mathscr{P}(X)$, the powerset of X. Finally, the total maps are the partial functions that are defined everywhere, and hence $\mathrm{Tot}(\mathbf{Pfn}) \cong \mathbf{Set}$.

Example 12. Let **Wstar** be the category of W^*-algebras and subunital normal positive linear maps (see [10, § 2.6] for the definitions). Then the opposite $\mathbf{Wstar}^{\mathrm{op}}$ is a σ-effectus with $\mathbb{C}$ as unit. A family of maps $f_j\colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Wstar}^{\mathrm{op}}$ for $j \in J$ is summable iff $\sum_{j \in F} f_j(1_{\mathfrak{B}}) \leq 1_{\mathfrak{A}}$ in $\mathfrak{A}$ for all finite $F \subseteq J$. Then define $(\bigovee_j f_j)(b) = \sum_{j \in J} f_j(b)$ where the infinite sum converges ultraweakly in $\mathfrak{A}$. States on $\mathfrak{A} \in \mathbf{Wstar}^{\mathrm{op}}$ are unital normal positive maps from $\mathfrak{A} \to \mathbb{C}$, which are known as *normal states* in the literature. The set of predicates $\mathrm{Pred}(\mathfrak{A}) = [0,1]_{\mathfrak{A}}$ is its unit interval. The total maps are precisely the unital maps. We note that the category of C*-algebras similarly forms an effectus, but not a σ-effectus [10, Example 7.3.36].

Remark 13. What we defined as an effectus is called an *effectus in partial form* in [11]. It is also possible to axiomatize an *effectus in total form*. Given an effectus C in partial form, the subcategory of total maps $\mathrm{Tot}(C)$ is an effectus in total form, which has a final object $1 = I$. As a total map $A \to B+1$ corresponds to a (partial) map $A \to B$, one can define from an effectus in total form a category of partial maps, which turns out to recover the original effectus in partial form. This correspondence leads to a 2-categorical equivalence of the relevant categories of effectuses [9] (see also [10, § 4.2]). We elected to work here with effectuses in partial form because the definition admits an obvious extension to the σ-additive case. One can define σ-effectuses in total form through the equivalence of the two form of effectuses, but we do not know whether they admit an intrinsic categorical characterization like effectuses in total form, which can be defined in terms of pullbacks and jointly monic morphisms [11, Definition 2].

By definition, predicates $p \colon A \to I$ in an effectus form an effect algebra. In a σ-effectus, predicates also have a σ-additive structure. We will show that the structure of predicates in a σ-effectus is captured precisely by the well-established notion of σ-effect algebras.

Definition 14. A **σ-effect algebra** [17, 21] is an effect algebra whose partial ordering is **ω-complete**, that is, where any increasing sequence $a_0 \le a_1 \le \dots$ has a supremum. We say a countable family $(x_j)_{j\in J}$ in a σ-effect algebra E is **summable** when the family $(x_j)_{j\in F}$ is summable for every finite subset $F \subseteq J$. For a summable countable family $(x_j)_{j\in J}$ we define $\bigovee_{j\in J} x_j = \bigvee_F \bigovee_{j\in F} x_j$ where F runs over all finite subsets of J, and the supremum exists by ω-completeness.

The definition of sums of countable families equips each σ-effect algebra with a canonical σ-PAM structure that extends its PCM structure. Conversely, each effect algebra that is a σ-PAM is ω-complete.

Proposition 15. *Let E be an effect algebra with a σ-PAM structure that extends the PCM structure of E. Then E is ω-complete and hence a σ-effect algebra. Moreover, the σ-PAM structure coincides with the canonical σ-PAM structure of the σ-effect algebra E.*

Proof. See Appendix A. $\square$

Corollary 16. *For any object A in a σ-effectus* $\mathbf{C}$*,* $\mathrm{Pred}(A) = \mathbf{C}(A, I)$ *forms a σ-effect algebra.* $\square$

The following, straightforwardly verifiable, lemma establishes the equivalence of two possible notions of morphisms of σ-effect algebras.

Lemma 17. *Let E, D be σ-effect algebras and $f \colon E \to D$ an additive map. Then f is σ-additive if and only if it is **ω-continuous**, i.e. if it preserves suprema of increasing sequence $a_0 \le a_1 \le \cdots$.* $\square$

3.1 Effect monoids and modules

The predicates of the unit object I in a (σ-)effectus do not just form a (σ-)effect algebra. As they are the morphisms $s : I \to I$ they also have a 'multiplication' operation given by composition of morphisms. The resulting structure in the finitary case is known as an effect monoid [24, 25]. We introduce σ-effect monoids as the counterpart for the countable case.

Definition 18. An **effect monoid** (resp. **σ-effect monoid**) is a (σ-)effect algebra $(M, \ovee, 0, 1)$ with an associative binary (total) operation $\cdot : M \times M \to M$ that is (σ-)biadditive and satisfies $a \cdot 1 = a = 1 \cdot a$ for all $a \in M$. Given an effect monoid M we define the **opposite** effect monoid M^{op} as the same underlying effect algebra, but with the product defined as $a \cdot' b \equiv b \cdot a$. Obviously M is commutative iff $M = M^{\mathrm{op}}$.

The monoids in the symmetric monoidal category of (σ-)effect algebras with (unital) morphisms and the algebraic tensor product are precisely the (σ-)effect monoids, hence the name [21,26].

The structure of ω-complete effect monoids has been studied in [37]. It follows from [37, Theorem 43] (with Lemma 17) that any ω-complete effect monoid is a σ-effect monoid — that is, the requirement of σ-biadditivity of the multiplication may be weakened to biadditivity.

Example 19. In **Pfn** the scalars are $\{0,1\}$, and hence $\{0,1\}$ is a σ-effect monoid. More generally, any Boolean algebra $(B,0,1,\wedge,\vee,(\ )^{\perp})$ (being an effect algebra by Example 7), is an effect monoid with $a\cdot b \equiv a\wedge b$. Therefore any ω-complete Boolean algebra is a σ-effect monoid.

Example 20. The scalars of $\mathbf{Wstar}^{\mathrm{op}}$ is the real unit interval $[0,1]$, which is thus a σ-effect monoid with the usual multiplication and partial addition. More generally, let X be a compact Hausdorff space. We denote its space of continuous functions into the complex numbers by $C(X) \equiv \{f : X \to \mathbb{C} \mid f \text{ continuous}\}$. This is a commutative unital C*-algebra (and conversely by the Gel'fand theorem, any commutative C*-algebra with unit is of this form). Its unit interval $[0,1]_{C(X)} = \{f : X \to [0,1] \mid f \text{ continuous}\}$ is not just an effect algebra but an effect monoid (with multiplication defined pointwise). The effect monoid $[0,1]_{C(X)}$ is ω-complete (and thus a σ-effect monoid) if and only if X is **basically disconnected**, i.e. when every cozero set has open closure [20, 1H & 3N.5].

These examples of effect monoids are all commutative. In [10, Ex. 4.3.9] and [39, Cor. 51] two different non-commutative effect monoids are constructed.

In the rest of this section, we study the structures of predicates and substates. In particular, it will be shown that any (σ-)effect monoid can appear as the scalars of a (σ-)effectus (Propositions 25 and 30).

For a monoid M, an M**-action** on a set X is a function $\cdot\colon M\times X\to X$ such that $1\cdot x = x$ and $(r\cdot s)\cdot x = r\cdot(s\cdot x)$ for all $r,s\in M$ and $x\in X$. We will apply this definition to (σ-)effect monoids.

Definition 21. Let M be a (σ-)effect monoid. A **(σ-)effect M-module** is a (σ-)effect algebra E equipped with a (σ-)biadditive M-action $\cdot\colon M\times E\to E$. Explicitly, for example, the biadditivity means:

$$(r\ovee s)\cdot a = r\cdot a\ovee s\cdot a \qquad r\cdot(a\ovee b) = r\cdot a\ovee r\cdot b \qquad 0\cdot a = 0 = r\cdot 0$$

for all $r,s\in M$ and $a,b\in E$ with $r\perp s$ and $a\perp b$. We write $\mathbf{EMod}_M$ (resp. $\sigma\mathbf{EMod}_M$) for the category of (σ-)effect M-modules and (σ-)additive maps that preserve the M-action; i.e. $f(r\cdot x) = r\cdot f(x)$.

Example 22. If $\mathbf{C}$ is a (σ-)effectus with scalars $M = \mathbf{C}(I,I)$, the set $\mathrm{Pred}(A)$ of predicates on $A\in\mathbf{C}$ is a (σ-)effect M-module, with M-action given by composition $r\cdot p = r\circ p$.

Example 23. A (σ-)effect $\{0,1\}$-module is just a (σ-)effect algebra, as the $\{0,1\}$-action is trivial.

Example 24. When M is the real unit interval $[0,1]$, an effect M-module is precisely a *convex effect algebra* [22]. These are effect algebras E that are intervals $[0,u]_V$ of ordered vector spaces V with a positive $u\in V$ [23]. We will come back to this in Section 5.2.

Proposition 25. *Let M be an effect monoid (resp. σ-effect monoid). Then the opposite category $\mathbf{EMod}_M^{\mathrm{op}}$ is an effectus (resp. $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ is a σ-effectus) with scalars M. The unit object is M, and coproducts are given by Cartesian products with pointwise operations (which form products in $\mathbf{EMod}_M$ and $\sigma\mathbf{EMod}_M$).*

Proof. See [10, Proposition 3.4.10] for the case of effect monoids. We prove the result for σ-effect monoids in Proposition 60 in Appendix A. □

This allows us to describe the assignment of predicates to each object as a morphism of effectuses.

Proposition 26. *Let* $\mathbf{C}$ *be an effectus (resp. σ-effectus) with scalars* $M = \mathbf{C}(I,I)$. *Then the assignment* $A \mapsto \mathrm{Pred}(A)$ *induces a morphism of effectuses* $\mathrm{Pred}\colon \mathbf{C} \to \mathbf{EMod}_M^{\mathrm{op}}$ *(resp. morphism of σ-effectuses* $\mathrm{Pred}\colon \mathbf{C} \to \sigma\mathbf{EMod}_M^{\mathrm{op}}$*).*

Proof. See [10, Lemma 4.2.11] for the case of effectuses. We prove the result for σ-effectuses in Proposition 61 in Appendix A. □

Now let us describe the 'dual' structure of the substates. Usually one focuses on the set of states, which forms an (abstract) M-convex set; see e.g. [11, 25, 38]. However, here we focus on the set of substates and axiomatize its structure as (σ-)weight M-modules. This is not just natural in the setting of effectuses in partial form, but also has the advantage that we can avoid technical problems with convex sets, see Remark 32 below.

Definition 27. Let M be a (σ-)effect monoid. A **(σ-)weight M-module** is a PCM (resp. σ-PAM) X equipped with a (σ-)biadditive M-action $\cdot\colon M \times X \to X$ and a function $|-|\colon X \to M$, called the **weight**, such that

- $|-|\colon X \to M$ is (σ-)additive and preserves the M-action, i.e. $|rx| = r|x|$;
- $|x| = 0$ implies $x = 0$;
- $|x| \perp |y|$ implies $x \perp y$ (resp. countable families $(x_j)_{j\in J}$ are summable when $(|x_j|)_{j\in J}$ is summable).

A function $f\colon X \to Y$ between (σ-)weight M-modules is **weight-preserving** if $|f(x)| = |x|$ for all $x \in X$, and **weight-decreasing** if $|f(x)| \leq |x|$ for all $x \in X$. We denote by $\mathbf{WMod}_M$ (resp. $\sigma\mathbf{WMod}_M$) the category of (σ-)weight M-modules and weight-decreasing (σ-)additive maps that preserves the M-action.

Example 28. If $\mathbf{C}$ is a (σ-)effectus with scalars $M = \mathbf{C}(I,I)$, the set $\mathrm{St}_{\leq}(A)$ of substates on $A \in \mathbf{C}$ is a (σ-)weight M^{op}-module, with M^{op}-action given by composition (from the right) $r \cdot \omega = \omega \circ r$, and weight $|\omega| = \mathbf{1} \circ \omega$. Note that states are precisely elements $\omega \in \mathrm{St}_{\leq}(A)$ with weight 1.

For a weight M-module X, let $B(X) = \{x \in X;\ |x| = 1\}$ be the set of elements with weight 1. The set $B(X)$ is closed under 'M-convex sums', i.e. $\bigovee_{i=1}^n r_i x_i \in B(X)$ for $x_i \in X$ and $r_i \in M$ with $\bigovee_i r_i = 1$. This makes $B(X)$ into an M-convex set [10, § 3.6]. In particular, the states $\mathrm{St}(A) = B(\mathrm{St}_{\leq}(A))$ in an effectus form an M-convex set. In this way, out treatment of substates subsumes the usual treatment of states in terms of convex sets. If M is 'well-behaved' such as when $M = [0,1]$, the category of M-convex sets is equivalent to the category of weight M-modules and *weight-preserving* maps [10, Proposition 4.4.10].

Example 29. Both weight $\{0,1\}$-modules and σ-weight $\{0,1\}$-modules are precisely **pointed sets**, i.e. sets X equipped with a distinguished element $x_0 \in X$. Every (σ-)weight $\{0,1\}$-module X is a pointed set $(X,0)$, and the converse is also true. This is because in a (σ-)weight $\{0,1\}$-module, all nonzero elements have weight 1 and thus they cannot be summable with nonzero elements. This yields isomorphisms of categories $\mathbf{WMod}_{\{0,1\}} \cong \sigma\mathbf{WMod}_{\{0,1\}} \cong \mathbf{Set}_*$, where $\mathbf{Set}_*$ denotes the category of pointed sets and functions that preserves the distinguished element.

Proposition 30. *Let M be an effect monoid (resp. σ-effect monoid). Then the category* $\mathbf{WMod}_M$ *is an effectus (resp.* $\sigma\mathbf{WMod}_M$ *is a σ-effectus) with scalars M. The unit object is M and coproducts are given by* $\coprod_{\lambda\in\Lambda} X_\lambda = \{(x_\lambda)_\lambda \in \prod_{\lambda\in\Lambda} X_\lambda;\ (|x_\lambda|)_{\lambda\in\Lambda}$ *is summable in* $M\}$ *for finite or countable* Λ.

Proof. See [10, Proposition 3.5.9] for the case of effect monoids. We prove the case of σ-effect monoids in Proposition 62 in Appendix A. □

Proposition 31. *Let* $\mathbf{C}$ *be a (σ-)effectus with scalars* $M = \mathbf{C}(I,I)$. *The assignment* $A \mapsto \mathrm{St}_{\leq}(A)$ *induces a morphism of effectuses* $\mathrm{St}_{\leq}\colon \mathbf{C} \to \mathbf{WMod}_{M^{\mathrm{op}}}$ *(resp. morphism of σ-effectuses* $\mathrm{St}_{\leq}\colon \mathbf{C} \to \sigma\mathbf{WMod}_{M^{\mathrm{op}}}$*.)*

Proof. See [10, Lemma 4.2.11] for the case of effectuses. We prove the result for σ-effectuses in Proposition 64 in Appendix A. ☐

Remark 32. Similar results to the previous two hold for M-convex sets and states in an effectus under certain additional assumptions on the effect monoid M and on the effectus; see [10, Corollary 4.4.15 and Proposition 4.5.11] and [38, § 3.2.4]. However, it is an open question whether the results hold in general.

4 Separation properties and normalization

The definition of a (σ-)effectus is quite weak. It will therefore be useful to consider some additional structure that an effectus might have. The first structure we consider is based on the notion of 'operational equivalence' used in GPTs (cf. [7, § 2.2]). This basically says that if two transformations act the same on all effects or substates that they must be the same transformations, since they are operationally indistinguishable.

Definition 33. A (σ-)effectus is **predicate-separated** when any pair of morphisms $f,g\colon A\to B$ satisfy $f=g$ whenever $p\circ f=p\circ g$ for all $p\in \mathrm{Pred}(B)$. It is **substate-separated** when any pair of morphisms $f,g\colon A\to B$ satisfy $f=g$ whenever $f\circ\omega=g\circ\omega$ for all substates $\omega\in \mathrm{St}_{\leq}(A)$.

The following is an immediate consequence from the definition, which will be used in Section 5.

Proposition 34. *A σ-effectus* $\mathbf{C}$ *is predicate-separated if and only if the morphism of σ-effectuses* $\mathrm{Pred}\colon \mathbf{C}\to\sigma\mathbf{EMod}_M^{\mathrm{op}}$ *is faithful (as a functor). It is substate-separated if and only if the morphism of σ-effectuses* $\mathrm{St}_{\leq}\colon \mathbf{C}\to\sigma\mathbf{WMod}_{M^{\mathrm{op}}}$ *is faithful.* ☐

Hence, a σ-effectus satisfying one of the separation properties can be seen as a 'sub-σ-effectus' of the σ-effectus of σ-effect modules or of σ-weight modules. One could argue that it would be more natural to assume state separation, instead of substate separation. An effectus is **state-separated** if for any pair of morphisms $f,g\colon A\to B$ we have $f=g$ whenever $f\circ\omega=g\circ\omega$ for all states $\omega\in\mathrm{St}(A)$. This however turns out to be equivalent to substate separation when the next condition we introduce is satisfied.

A second property that is usually assumed (often implicitly) in a GPT is the possibility of normalizing states (cf. [7, § 4.1.4], [12, § 5.4.1]). A 'normalized' state ω is one that has unit probability when the deterministic effect ('always true') is tested against it: $\mathbf{1}\circ\omega=1$. An 'unnormalized' substate can then be interpreted as one that has a probability of failure at being prepared: $\mathbf{1}\circ\omega<1$. Being able to normalize a state recognizes the possibility of deterministically preparing any state that can be probabilistically prepared.

Definition 35. A (σ-)effectus admits **normalization** if for each nonzero substate $\omega\colon I\to A$, there exists a unique state $\bar{\omega}\colon I\to A$ such that $\omega=\bar{\omega}\circ(\mathbf{1}\circ\omega)$.

Proposition 36. *A (σ-)effectus with normalization is state-separated if and only if it is substate-separated.*

Proof. See Appendix B. ☐

In [9, Proposition 6.4], it was shown that if an effectus admits normalization, the scalars admit a type of division. In a σ-effectus, the converse holds, together with several other equivalent conditions.

Theorem 37. *Let* $\mathbf{C}$ *be a σ-effectus. The following are equivalent.*

(i) $\mathbf{C}$ *admits normalization.*

(ii) *The effect monoid* $\mathbf{C}(I,I)$ *admits* ***division****: for any* $s,t \in \mathbf{C}(I,I)$ *with* $s \leq t$ *and* $t \neq 0$*, there is a unique* $s/t \in \mathbf{C}(I,I)$ *satisfying* $(s/t) \cdot t = s$.

(iii) *The effect monoid* $\mathbf{C}(I,I)$ *has no nontrivial zero divisors, i.e.* $s \cdot t = 0$ *implies* $s = 0$ *or* $t = 0$.

(iv) *Every nonzero scalar* $s\colon I \to I$ *in* $\mathbf{C}$ *is an epi.*

Proof. See Appendix B. □

5 Classification of σ-effectuses with normalization

In this section, we combine the theory of σ-effectuses with the classification result of ω-complete effect monoids obtained in [37]. It leads to the classification of σ-effectuses with normalization: these σ-effectuses are either the trivial category, σ-effectuses with Boolean scalars $\{0,1\}$, or σ-effectuses with probabilistic scalars $[0,1]$. We then investigate the latter two cases in more detail, assuming the separation properties.

In Examples 19 and 20 we presented two examples of ω-complete effect monoids: ω-complete Boolean algebras and $[0,1]_{C(X)}$ for basically disconnected compact Hausdorff spaces X. One of the main results of [37] shows that these examples are basically the only possible ω-complete effect monoids.

Theorem 38 ([37, Theorem 54])**.** *Let* M *be an* ω*-complete effect monoid. Then* M *embeds into* $M_1 \oplus M_2$*, where* M_1 *is an* ω*-complete Boolean algebra, and* $M_2 = [0,1]_{C(X)}$*, where* X *is a basically disconnected compact Hausdorff space.* □

It immediately follows that any ω-complete effect monoid is commutative, since both M_1 and M_2 above are commutative. Hence we obtain the following result.

Corollary 39. *The scalars of a* σ*-effectus are commutative.* □

Theorem 38 has the following consequence, also shown in [37].

Theorem 40 ([37, Theorem 71])**.** *Let* M *be an* ω*-complete effect monoid with no non-trivial zero divisors. Then either* $M = \{0\}$*,* $M = \{0,1\}$ *or* $M = [0,1]$. □

Combining Theorems 40 and 37 we immediately get the following result characterizing the possible scalars in a σ-effectus with normalization.

Theorem 41. *A* σ*-effectus* $\mathbf{C}$ *admits normalization if and only if the effect monoid* $\mathbf{C}(I,I)$ *of scalars is isomorphic to* $\{0\}$*,* $\{0,1\}$*, or* $[0,1]$. □

Of these three options, the first always leads to a trivial effectus.

Proposition 42. *Let* $\mathbf{C}$ *be an effectus where the scalars* $\mathbf{C}(I,I)$ *are isomorphic to* $\{0\}$*. Then* $\mathbf{C}$ *is equivalent to the trivial category with a single object and a single morphism.*

Proof. Because $\mathrm{id} = 0\colon I \to I$, any truth map $\mathbf{1}\colon A \to I$ satisfies $\mathbf{1} = \mathrm{id} \circ \mathbf{1} = 0 \circ \mathbf{1} = 0$. Thus for any morphism $f\colon A \to B$ we have $\mathbf{1} \circ f = 0 \circ f = 0$. By an axiom of effectuses, we obtain $f = 0$. Therefore for any objects $A, B \in \mathbf{C}$, the homset $\mathbf{C}(A,B)$ is a singleton. We conclude that $\mathbf{C}$ is equivalent to the trivial category. □

5.1 σ-Effectus with Boolean scalars

If a σ-effectus $\mathbf{C}$ has Boolean scalars $\{0,1\}$, the operational theory described by $\mathbf{C}$ is deterministic: every predicate either holds with certainty on each state, or does not hold at all. Therefore such an effectus is fundamentally classical, as it is well-known that quantum theory cannot be described as a deterministic theory.

Example 43. Let $\sigma\mathbf{EA}$ be the category of σ-effect algebras and σ-additive maps. We have $\sigma\mathbf{EA} \cong \sigma\mathbf{EMod}_{\{0,1\}}$, and hence $\sigma\mathbf{EA}^{\mathrm{op}}$ is an σ-effectus with scalars $\{0,1\}$. Therefore $\sigma\mathbf{EA}^{\mathrm{op}}$ is deterministic and 'classical'. It may seem to contradict the fact that σ-effect algebras also include spaces of quantum effects. This paradoxical situation can be explained as follows.

Let H be a Hilbert space with $\dim(H) > 2$, and let $E = [0,1]_{B(H)}$ be the set of effects on H (see Example 8). Then E is a σ-effect algebra. The subset of projections $P(H) \subseteq E$ is then an σ-effect subalgebra and hence is an object in the effectus $\sigma\mathbf{EA}^{\mathrm{op}}$. By the Kochen–Specker theorem [28], we have $\mathrm{St}(P(H)) \equiv \mathrm{Tot}(\sigma\mathbf{EA}^{\mathrm{op}})(\{0,1\},P(H)) = \emptyset$, that is, there exists no unital σ-additive map $P(H) \to \{0,1\}$. This implies $\mathrm{St}(E) = \emptyset$ too. Operationally speaking, therefore, one cannot prepare a system of type $P(H)$ or E in $\sigma\mathbf{EA}^{\mathrm{op}}$. In other words, both $P(H)$ and E are operationally equivalent to the empty system 0.

This observation motivates us to restrict ourselves to σ-effectuses with scalars $\{0,1\}$ that are substate-separated (or equivalently, state-separated, by Proposition 36), in order to take operational equivalence into account. We will show that these σ-effectuses always embed into the σ-effectus $\mathbf{Pfn}$ of sets and partial functions via faithful morphisms of σ-effectuses, and hence they are 'sub-σ-effectuses' of $\mathbf{Pfn}$. We also show that they embed into the σ-effectus of ω-complete Boolean algebras. These results make it more precise what we mean by 'σ-effectuses with scalars $\{0,1\}$ are classical'.

Proposition 44. *We have an equivalence of categories $\sigma\mathbf{WMod}_{\{0,1\}} \xrightarrow{\simeq} \mathbf{Pfn}$. The functor is also a morphism of σ-effectuses.*

Proof. As we observed in Example 29, σ-weight $\{0,1\}$-modules are merely pointed sets: $\sigma\mathbf{WMod}_{\{0,1\}} \cong \mathbf{Set}_*$. Then the equivalence of the categories $\mathbf{Set}_* \simeq \mathbf{Pfn}$ is well-known — it sends $f\colon (X,x_0) \to (Y,y_0)$ in $\mathbf{Set}_*$ to $\overline{f}\colon X\setminus\{x_0\} \to Y\setminus\{y_0\}$ in $\mathbf{Pfn}$ where $\overline{f}(x)$ is defined iff $f(x) \neq y_0$ and in that case $\overline{f}(x) = f(x)$. The equivalence $\sigma\mathbf{WMod}_{\{0,1\}} \cong \mathbf{Set}_* \simeq \mathbf{Pfn}$ preserves all coproducts, and it is easily checked that it preserves the unit object. Hence it is also a morphism of σ-effectuses. $\square$

Combining it with Proposition 34, and with easy calculation, we obtain the following theorem.

Theorem 45. *Let $\mathbf{C}$ be a substate-separated σ-effectus with $\mathbf{C}(I,I) \cong \{0,1\}$. Then there is a faithful morphism of σ-effectuses $F\colon \mathbf{C} \to \mathbf{Pfn}$. Moreover, we have $\mathrm{St}(A) \cong FA$ for all $A \in \mathbf{C}$.* $\square$

We write $\omega\mathbf{BA}$ for the category of ω-complete[2] Boolean algebras and functions that preserves countable joins and *nonempty* countable meets. Then one can show that $\omega\mathbf{BA}^{\mathrm{op}}$ is a σ-effectus — in fact, $\omega\mathbf{BA}$ is a full subcategory of $\sigma\mathbf{EA}$ (the fullness is proved similarly to [10, Lemma 6.5.18]). The following result can be easily verified.

Proposition 46. *The contravariant powerset functor is a faithful morphism of σ-effectuses $\mathscr{P}\colon \mathbf{Pfn} \to \omega\mathbf{BA}^{\mathrm{op}}$, where $\mathscr{P}(f)(S) = \{x \in X \mid f(x) \text{ is defined and } f(x) \in S\}$ for partial functions $f\colon X \rightharpoonup Y$ and $S \in \mathscr{P}(Y)$.* $\square$

Composition of these last two faithful morphisms of σ-effectuses yields the following result.

[2] For a Boolean algebra, ω-completeness is equivalent to existence of all countable joins (and meets).

Theorem 47. *Let* $\mathbf{C}$ *be a substate-separated* σ*-effectus with scalars* $\{0,1\}$*. Then there is a faithful morphism of* σ*-effectuses* $G\colon \mathbf{C} \to \omega\mathbf{BA}^{\mathrm{op}}$. □

This does *not* mean that the predicates $\mathrm{Pred}(A)$ form a Boolean algebra, but rather there is an injection

$$\mathrm{Pred}(A) \equiv \mathbf{C}(A,I) \rightarrowtail \omega\mathbf{BA}^{\mathrm{op}}(GA,GI) \cong \omega\mathbf{BA}(\{0,1\},GA) \cong GA,$$

so that predicates form a subset of the Boolean algebra GA. In fact, we can prove that the injection $\mathrm{Pred}(A) \rightarrowtail GA$ is a σ-additive map. From this it follows that $\mathrm{Pred}(A)$ is an *orthoalgebra*, i.e. that it has the property that $p \perp p$ implies $p = 0$.

5.2 σ-Effectus with probabilistic scalars

In this section we will show that a σ-effectus with scalars $[0,1]$ can be embedded into the categories of certain ordered vector spaces, under the assumption of the separation properties. These ordered vector spaces are order-unit spaces and (pre-)base-norm spaces, which serve as abstract spaces of effects and of states, respectively. They have long been used in GPT-style approaches to quantum theory (also known as 'convex operational' approaches); see e.g. [13, 14, 29, 30] and recent work [3, 6, 18, 19].

The embedding results are obtained as consequences of representation results of σ-effect $[0,1]$-modules and (cancellative) σ-weight $[0,1]$-modules into suitable order-unit spaces and (pre-)base-norm spaces. For space reasons, the proofs of Propositions 52, 55, and 56 are deferred to Appendix C.

We start by recalling the known representation result of effect $[0,1]$-modules.

Definition 48. Let A be an ordered vector space (with positive cone A_+). An **order unit** of A is a positive element $u \in A_+$ such that for all $x \in A$ there exists $n \in \mathbb{N}$ with $-nu \le x \le nu$. A map $f\colon A \to B$ between ordered vector spaces with order unit (say $u_A \in A$ and $u_B \in B$) is **subunital** if $f(u_A) \le u_B$. We write **OVSu** for the category of ordered vector spaces with order unit and subunital positive linear maps. (A map $f\colon A \to B$ is **positive** if $f(A_+) \subseteq B_+$.)

Note that for each $(A,u) \in \mathbf{OVSu}$, the unit interval $[0,u]_A = \{a \in A \mid 0 \le a \le u\}$ is an effect $[0,1]$-module. Conversely, for each effect $[0,1]$-module E, one can construct $(A,u) \in \mathbf{OVSu}$ such that $[0,u]_A \cong E$ [23, 27]. These constructions yield an equivalence of categories.

Proposition 49 ([27, Theorem 14])**.** *The functor* $\mathbf{OVSu} \to \mathbf{EMod}_{[0,1]}$ *that sends* (A,u) *to* $[0,u]_A$ *is an equivalence of categories.* □

Definition 50. An **order-unit space** is an ordered vector space A with order unit u satisfying the Archimedean property: $nx \le u$ for all $n \in \mathbb{N}$ implies $x \le 0$. Each order-unit space (A,u) is equipped with the intrinsic **order-unit norm** given by $\|a\| = \inf\{r > 0 \mid -ru \le a \le ru\}$. A **Banach order-unit space** is an order-unit space that is complete with respect to the order-unit norm.

Definition 51. An ordered vector space A is **monotone σ-complete** if every ascending sequence $a_0 \le a_1 \le \cdots$ in A that is bounded above has a supremum $\bigvee_{n=0}^{\infty} a_n$. A map between monotone σ-complete ordered vector spaces is **σ-normal** if it preserves suprema of ascending sequences that are bounded above. We write σ**BOUS** for the category of monotone σ-complete Banach order-unit spaces and σ-normal subunital positive linear maps.

The equivalence of Proposition 49 can be restricted to the following one.

Proposition 52. *There is an equivalence of categories* $\sigma\mathbf{BOUS} \simeq \sigma\mathbf{EMod}_{[0,1]}$.

This proves that $\sigma\mathbf{BOUS}^{\mathrm{op}}$ is a σ-effectus. By Proposition 34, we obtain the following result.

Theorem 53. *Let* $\mathbf{C}$ *be a predicate-separated* σ*-effectus with scalars* $\mathbf{C}(I,I) \cong [0,1]$. *Then there is a faithful morphism of* σ*-effectuses* $F\colon \mathbf{C} \to \sigma\mathbf{BOUS}^{\mathrm{op}}$. *Furthermore,* $\mathrm{Pred}(A) \cong [0,u]_{FA}$ *for all* $A \in \mathbf{C}$. □

While this representation onto vector spaces uses the structure of the predicates in the effectus, we can dually find a representation using the structure of the states. For this we will need a representation of (σ-)weight $[0,1]$-modules.

Definition 54. An **ordered vector space with trace**[3] is an ordered vector space V that is positively generated (i.e. $V = V_+ - V_+$) and equipped with a linear functional $\tau\colon V \to \mathbb{R}$ called the **trace** that is **strictly positive** in the sense that $x > 0$ implies $\tau(x) > 0$. A map $f\colon V \to W$ between ordered vector spaces with trace is **trace-decreasing** if $\tau_W(f(x)) \leq \tau_V(x)$ for all $x \in V_+$. We write **OVSt** for the category of ordered vector spaces with trace and trace-decreasing positive linear maps.

Each $(V,\tau) \in \mathbf{OVSt}$ defines a weight $[0,1]$-module via its **subbase** $B_{\leq}(V) = \{x \in V_+ \mid \tau(x) \leq 1\}$, with weight $|x| = \tau(x)$. Clearly, $B_{\leq}(V)$ is **cancellative** in the sense that $x \ovee y = x \ovee z$ implies $y = z$. Writing $\mathbf{CWMod}_{[0,1]} \hookrightarrow \mathbf{WMod}_{[0,1]}$ for the full subcategory of cancellative weight $[0,1]$-modules, we obtain a functor $B_{\leq}\colon \mathbf{OVSt} \to \mathbf{CWMod}_{[0,1]}$. Conversely, for any cancellative weight $[0,1]$-module X we can construct $V \in \mathbf{OVSt}$ such that $B_{\leq}(V) \cong X$, giving rise to an equivalence of categories.

Proposition 55. *The functor* $B_{\leq}\colon \mathbf{OVSt} \to \mathbf{CWMod}_{[0,1]}$ *is an equivalence of categories.*

Each $(V,\tau) \in \mathbf{OVSt}$ is equipped with an intrinsic seminorm given by:

$$\|x\| \;=\; \inf\{\tau(x_1) + \tau(x_2) \mid x_1, x_2 \in V_+ \;\text{ such that }\; x = x_1 - x_2\}\,.$$

Following Furber [18], we call (V,τ) a **pre-base-norm space** if the seminorm $\|-\|$ is a norm (i.e. $\|x\| = 0$ implies $x = 0$). It is a **Banach pre-base-norm space** if V is complete with respect to the base norm. To formulate the results below, we introduce additional (non-standard) terminology. A Banach pre-base-norm space has a σ**-closed subbase** if for each countable family $(x_n)_{n\in\mathbb{N}}$ in $B_{\leq}(V)$ with $\sum_{n\in\mathbb{N}} \tau(x_n) \leq 1$, the series $\sum_{n=0}^{\infty} x_n$ converges to an element in $B_{\leq}(V)$.[4]

We write $\sigma\mathbf{BBNS} \hookrightarrow \mathbf{OVSt}$ for the full subcategory of Banach pre-base-norm spaces with a σ-closed subbase, and $\sigma\mathbf{CWMod}_{[0,1]} \hookrightarrow \sigma\mathbf{WMod}_{[0,1]}$ for the full subcategory of cancellative σ-weight $[0,1]$-modules. The equivalence of Proposition 55 can be restricted to these categories.

Proposition 56. *There is an equivalence of categories* $\sigma\mathbf{BBNS} \simeq \sigma\mathbf{CWMod}_{[0,1]}$.

As $\sigma\mathbf{CWMod}_{[0,1]}$ is a full subcategory of $\sigma\mathbf{WMod}_{[0,1]}$, it is a σ-effectus, and hence so is $\sigma\mathbf{BBNS}$. Combining Propositions 56 and 34 we have the following result.

Theorem 57. *Let* $\mathbf{C}$ *be a state-separated* σ*-effectus with scalars* $[0,1]$ *such that substates* $\mathrm{St}_{\leq}(A)$ *are cancellative. Then there is a faithful morphism of* σ*-effectuses* $G\colon \mathbf{C} \to \sigma\mathbf{BBNS}$. *Furthermore,* $\mathrm{St}_{\leq}(A) \cong B_{\leq}(GA)$ *for all* $A \in \mathbf{C}$. □

Remark 58. Cancellativity of the substates follows when the effectus is predicate-separated, and hence any state- and predicate-separated σ-effectus with scalars $[0,1]$ embeds into both $\sigma\mathbf{BBNS}$ and $\sigma\mathbf{BOUS}^{\mathrm{op}}$.

6 Conclusion

We introduced the notion of a σ-effectus and showed that when they allow normalization of states, the scalars must be equal to $\{0\}$, $\{0,1\}$, or $[0,1]$. The first case was shown to lead to a trivial effectus. In the

[3] It is called a *base ordered linear space* in [34] and a *semi-base-norm space* in [10].

[4] This property is equivalent to the assumption of the theorem of Edwards and Gerzon [15].

latter two cases we found that when operationally motivated state- and/or predicate-separation properties are satisfied, in the $\{0,1\}$ case the effectus embeds into the category of sets and partial functions, and thus is classical and deterministic, while in the $[0,1]$ case σ-effectuses embed into either a category of Banach order-unit spaces, or of Banach pre-base-norm spaces. We hence have found a dichotomy between deterministic and probabilistic models of physical theories from abstract categorical considerations.

For future work it might be interesting to consider what can be said about σ-effectuses when the normalization condition is dropped, which would allow for more complex scalars that can also represent 'spatial' systems as in [33].

A further open problem that needs to be addressed is whether the nice categorical definition of an effectus in total form can be modified to give a notion of an 'σ-effectus in total form' (see Remark 13). If this is the case, then our results imply a natural categorical characterization of Banach order-unit and pre-base-norm spaces.

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A Proofs in Section 3

Proof of Proposition 15. We write $\tilde{\bigovee}$ for the given σ-PAM operation on E. Let $a_0 \le a_1 \le \cdots$ be an increasing sequence in E. Let $b_0 = a_0$ and $b_{n+1} = a_{n+1} \ominus a_n$ for each $n \in \mathbb{N}$. Then we have $a_n = \bigovee_{k\le n} b_k$, and in particular, every finite subfamily of $(b_n)_{n\in\mathbb{N}}$ is summable. Therefore the sum $\tilde{\bigovee}_{n\in\mathbb{N}} b_n$ exists. We will prove that $\tilde{\bigovee}_{n\in\mathbb{N}} b_n$ is a supremum of $(a_n)_n$. We have

$$\tilde{\bigovee_{n\in\mathbb{N}}} b_n \;=\; \Big(\bigovee_{k\le n} b_n\Big) \ovee \Big(\tilde{\bigovee_{k>n}} b_k\Big) \;=\; a_n \ovee \Big(\tilde{\bigovee_{k>n}} b_k\Big),$$

so that $\tilde{\bigovee}_{n\in\mathbb{N}} b_n$ is an upper bound of $(a_n)_n$. Suppose that c is an upper bound of $(a_n)_n$. Then $\bigovee_{k\le n} b_k \le c$ for any $n \in \mathbb{N}$, and hence the sequence $c^\perp, b_0, \ldots, b_n$ is summable for any $n \in \mathbb{N}$. This implies that the sum $c^\perp \ovee (\tilde{\bigovee}_{n\in\mathbb{N}} b_n)$ exists. Hence $\tilde{\bigovee}_{n\in\mathbb{N}} b_n \le c$, as desired. Therefore E is ω-complete. To verify that $\tilde{\bigovee}$ coincides with the canonical σ-PAM structure, let $(x_j)_{j\in J}$ be a summable countable family. If J is finite, it is clear that $\tilde{\bigovee}_j x_j = \bigovee_j x_j$. If J is infinite, then we may assume $J = \mathbb{N}$ without loss of generality. Then the same argument as above proves $\tilde{\bigovee}_{n\in\mathbb{N}} x_n = \bigvee_{n\in\mathbb{N}} \bigovee_{k\le n} x_k$, and the right-hand side coincides with canonical $\bigovee_{n\in\mathbb{N}} x_n$. □

To prove that $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ and $\sigma\mathbf{WMod}_M$ are σ-effectuses, we use the following characterization of σ-effectuses (cf. a characterization of σ-PACs given in [1, § 5]).

Lemma 59. *Let $\mathbf{C}$ be a category with a distinguished object I and a family of maps $\mathbf{1}_A : A \to I$. Then $(\mathbf{C}, I)$ forms an σ-effectus with truth maps $\mathbf{1}_A : A \to I$ if and only if the following hold.*

(i) $\mathbf{C}$ *has countable coproducts.*

(ii) $\mathbf{C}$ *has zero morphisms.*

(iii) *For each object A and each countable set J, the partial projections $\rhd_j : J \cdot A \to A$ from the copower of A by J (i.e. the J-fold coproduct) are jointly monic.*

(iv) *Let $(f_j : A \to B)_{j\in J}$ be a countable family of parallel morphisms. If the family $(f_j : A \to B)_{j\in F}$ is compatible for each finite subset $F \subseteq J$, then $(f_j : A \to B)_{j\in J}$ is compatible.*

(v) $\mathbf{1}_{A+B} = [\mathbf{1}_A, \mathbf{1}_B] : A + B \to I$ *for all A, B.*

(vi) $\mathbf{1}_B \circ f = 0_{AI}$ *implies $f = 0_{AB}$ for all $f : A \to B$.*

(vii) *For all $f, g : A \to B$, if $\mathbf{1}_B \circ f, \mathbf{1}_B \circ g : A \to I$ are compatible, then f, g are compatible too.*

(viii) *For each* $p\colon A \to I$, *there exists a unique* $p^{\perp}\colon A \to I$ *such that* $p, p^{\perp}$ *are compatible and* $\nabla_I \circ \langle\langle p, p^{\perp}\rangle\rangle = \mathbf{1}_A$, *where* $\nabla_I\colon I+I \to I$ *is the codiagonal and* $\langle\langle p, p^{\perp}\rangle\rangle\colon A \to I+I$ *is a unique (by* (iii)*) map satisfying* $\rhd_1 \circ \langle\langle p, p^{\perp}\rangle\rangle = p$ *and* $\rhd_2 \circ \langle\langle p, p^{\perp}\rangle\rangle = p^{\perp}$.

Proof. It is easy to verify the 'only if' direction. Conversely, when $\mathbf{C}$ satisfies (i)–(viii), we define addition on morphisms as follows. A countable family of morphisms $(f_j\colon A \to B)$ is summable iff it is compatible. In that case, by the joint monicity condition (iii), there is a unique morphism $f\colon A \to J \cdot B$ such that $f_j = \rhd_j \circ f$ for all $j \in J$. Then we define the sum by $\bigovee_{j\in J} f_j \coloneqq \nabla \circ f$, where $\nabla\colon J \cdot B \to B$ is the codiagonal. It is not hard to verify that this addition on each homset satisfies the axioms of σ-PAMs, σ-PACs, and σ-effectuses. The details can be found in [10, Proposition 3.8.6 and Lemma 7.3.38]. □

Proposition 60. *Let* M *be a* σ*-effect monoid. Then the opposite category* $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ *is a* σ*-effectus.*

Proof. We invoke Lemma 59. We take $I = M$ and $\mathbf{1}_E\colon E \to M$ in $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ to be the map $\mathbf{1}_E\colon M \to E$ in $\sigma\mathbf{EMod}_M$ given by $\mathbf{1}_E(s) = s \cdot 1$

(i) $\sigma\mathbf{EMod}_M$ has all products given by Cartesian products $\prod_j E_j$ with operations defined pointwise. Thus $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ has all coproducts.

(ii) The constant zero functions are zero morphisms in $\sigma\mathbf{EMod}_M$, and hence in $\sigma\mathbf{EMod}_M^{\mathrm{op}}$.

(iii) Let J be a countable set. The partial projections $\rhd_j\colon J \cdot E \to E$ in $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ are morphisms $\rhd_j\colon E \to E^J$ in $\sigma\mathbf{EMod}_M$ that send $x \in E$ to the J-tuple that has 0 at every coordinate except x at the jth coordinate. If $f, g\colon E^J \to D$ in $\sigma\mathbf{EMod}_M$ satisfy $f \circ \rhd_j = g \circ \rhd_j$ for all $j \in J$, then

$$f((x_j)_j) = f(\textstyle\bigovee_j \rhd_j(x_j)) = \bigovee_j f(\rhd_j(x_j)) = \bigovee_j g(\rhd_j(x_j)) = \cdots = g((x_j)_j).$$

Therefore the maps $\rhd_j$ are jointly epic in $\sigma\mathbf{EMod}_M$ and hence jointly monic in the opposite.

(iv) We prove that a countable family $(f_j\colon E \to D)_{j\in J}$ in $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ is compatible if and only if $(f_j(1))_{j\in J}$ is summable in E. By the limit axiom in E, this implies (iv) of Lemma 59. Let $(f_j)_{j\in J}$ be a compatible family. Then in $\sigma\mathbf{EMod}_M$, there exists a map $f\colon D^J \to E$ such that $f \circ \rhd_j = f_j$. Since $(1)_{j\in J} \in D^J$ can be written as $\bigovee_{j\in J} \rhd_j(1)$, it follows that the sum $\bigovee_{j\in J} f_j(1) = f(\bigovee_{j\in J} \rhd_j(1))$ is defined. Conversely, if $(f_j(1))_{j\in J}$ is summable, define a map $\langle\langle f_j\rangle\rangle_j\colon D^J \to E$ by $\langle\langle f_j\rangle\rangle_j((a_j)_j) = \bigovee_j f_j(a_j)$. We can prove that $\langle\langle f_j\rangle\rangle_j\colon D^J \to E$ is a morphism in $\sigma\mathbf{EMod}_M$. Then $(f_j)_{j\in J}$ is compatible via $\langle\langle f_j\rangle\rangle_j$.

(v) $\langle \mathbf{1}_E, \mathbf{1}_D\rangle(s) = (\mathbf{1}_E(s), \mathbf{1}_D(s)) = (s\cdot 1, s\cdot 1) = s\cdot(1,1) = \mathbf{1}_{E\times D}(1)$ in $\sigma\mathbf{EMod}_M$ and thus $[\mathbf{1}_E, \mathbf{1}_D] = \mathbf{1}_{E+D}$ in $\sigma\mathbf{EMod}_M^{\mathrm{op}}$.

(vi) Let $f\colon E \to D$ be a morphism with $\mathbf{1}_D \circ f = 0$ in $\sigma\mathbf{EMod}_M^{\mathrm{op}}$. It is a morphism $f\colon D \to E$ in $\sigma\mathbf{EMod}_M$, which satisfies $0 = 0(1) = (f \circ \mathbf{1}_D)(1) = f(1)$. Then for any $a \in D$ we have $0 \le f(a) \le f(1) = 0$ and therefore f is the constant zero function.

(vii) Let $f, g\colon E \to D$ be morphisms in $\sigma\mathbf{EMod}_M^{\mathrm{op}}$ such that $\mathbf{1}_D \circ f$ and $\mathbf{1}_D \circ g$ are compatible. By the characterization of the compatibility in point (iv), $f(1) = (f \circ \mathbf{1}_D)(1)$ and $g(1) = (g \circ \mathbf{1}_D)(1)$ are summable in E. Again by this characterization, f and g are compatible.

(viii) This holds because $p \mapsto p(1)$ defines a bijection $\sigma\mathbf{EMod}_M(M, E) \cong E$ that sends $\mathbf{1}_E\colon M \to E$ to $1 \in E$ and preserves sums $\ovee$, where the sums in $\mathrm{Pred}(E)$ are defined by $p \ovee q \coloneqq \nabla_I \circ \langle\langle p, q\rangle\rangle$. □

Proposition 61. *Let* $\mathbf{C}$ *be an effectus with scalars* $M = \mathbf{C}(I, I)$. *Then the assignment* $A \mapsto \mathrm{Pred}(A)$ *induces a morphism of* σ*-effectuses* $\mathrm{Pred}\colon \mathbf{C} \to \sigma\mathbf{EMod}_M^{\mathrm{op}}$.

Proof. The well-definedness of the functor $\mathrm{Pred}\colon \mathbf{C} \to \sigma\mathbf{EMod}_M^{\mathrm{op}}$ is easy. It preserves the unit object: we have $\mathrm{Pred}(I) = \mathbf{C}(I,I) = M$ and $\mathrm{Pred}(\mathbf{1}_A) = (-)\circ \mathbf{1}_A = \mathbf{1}_{\mathrm{Pred}(A)}$. It sends countable coproducts in $\mathbf{C}$ to products in $\sigma\mathbf{EMod}_M$:

$$\mathrm{Pred}(\coprod_\lambda E_\lambda) \;=\; \mathbf{C}(\coprod_\lambda E_\lambda, I) \cong \prod_\lambda \mathbf{C}(E_\lambda, I) \;=\; \prod_\lambda \mathrm{Pred}(E_\lambda)\,.$$

It is easy to see that the bijection is indeed an isomorphism in $\sigma\mathbf{EMod}_M$. □

Proposition 62. *Let M be an σ-effect monoid. Then the category $\sigma\mathbf{WMod}_M$ is a σ-effectus.*

Proof. We invoke Lemma 59. We take $I = M$ and define $\mathbf{1}_X : X \to M$ in $\sigma\mathbf{WMod}_M$ by $\mathbf{1}_X(x) = |x|$.

(i) First we show that $\sigma\mathbf{WMod}$ has countable coproducts. For a countable family $(X_\lambda)_{\lambda\in\Lambda}$ of objects, we define the underlying set by

$$\coprod_{\lambda\in\Lambda} X_\lambda \;=\; \Big\{(x_\lambda)_\lambda \in \prod_{\lambda\in\Lambda} X_\lambda \;\Big|\; (\,|x_\lambda|\,)_{\lambda\in\Lambda} \text{ is summable in } M\Big\}$$

and the weight of $(x_\lambda)_\lambda \in \coprod_{\lambda\in\Lambda} X_\lambda$ by $|(x_\lambda)_\lambda| = \bigovee_{\lambda\in\Lambda} |x_\lambda|$. This determines summability in $\coprod_{\lambda\in\Lambda} X_\lambda$: a countable family $((x_{\lambda j})_\lambda)_{j\in J}$ is summable if $(\,|(x_{\lambda j})_\lambda|\,)_{j\in J} = (\bigovee_\lambda |x_{\lambda j}|)_{j\in J}$ is summable in M. We define the σ-PAM structure and and M-action pointwise. It is straightforward to verify that $\coprod_{\lambda\in\Lambda} X_\lambda$ is a σ-weight module, and that it is a coproduct with coprojections $\kappa_\lambda : X_\lambda \to \coprod_{\lambda\in\Lambda} X_\lambda$ that sends each element $x \in X_\lambda$ to the Λ-tuple with 0 everywhere except x at the λth coordinate.

(ii) The constant zero functions $0\colon X \to Y$ form zero morphisms in $\sigma\mathbf{WMod}_M$.

(iii) The partial projections $\rhd_k\colon J\cdot X \to X$ for $k \in J$ are given by $\rhd_k((x_j)_j) = x_k$. It is clear that these maps are jointly monic.

(iv) Let $(f_j\colon X \to Y)_{j\in J}$ be a countable family of morphisms in $\sigma\mathbf{WMod}_M$. We claim that $(f_j)_j$ is compatible if and only if $\bigovee_j |f_j(x)|$ is defined and $\bigovee_j |f_j(x)| \le |x|$ for all $x \in X$. This implies (iv) of Lemma 59, because $\bigovee_j |f_j(x)|$ is the supremum of the sums $\bigovee_{j\in F} |f_j(x)|$ for finite subsets $F \subseteq J$. Suppose that $(f_j)_j$ is compatible via $f\colon X \to J\cdot Y$. Then for each $x \in X$, one has $\rhd_j(f(x)) = f_j(x)$, and thus by definition of $\rhd_j$, we have $f(x) = (f_j(x))_j$. As f is weight-decreasing,

$$|x| \;\ge\; |f(x)| \;=\; |(f_j(x))_j| \;=\; \bigovee_j |f_x(x)|\,.$$

Conversely, if $\bigovee_j |f_j(x)| \le |x|$ for all $x \in X$, then we can show that the map $f\colon X \to J\cdot Y$ given by $f(x) = (f_j(x))_j$ is a well-defined morphism in $\sigma\mathbf{WMod}_M$ and that $(f_j)_{j\in J}$ is compatible via f.

(v) $\mathbf{1}_{X+Y}(x,y) = |(x,y)| = |x| \ovee |y| = \mathbf{1}_X(x) \ovee \mathbf{1}_Y(y) = [\mathbf{1}_X, \mathbf{1}_Y](x,y)$.

(vi) Suppose that $f\colon X \to Y$ satisfies $\mathbf{1}_Y \circ f = 0$. For each $x \in X$, we then have $|f(x)| = 0$ and hence $f(x) = 0$. Therefore $f = 0$.

(vii) Let $f,g\colon X \to Y$ be morphisms such that $\mathbf{1}_Y \circ f$ and $\mathbf{1}_Y \circ g$ are compatible. By the characterization of the compatibility in point (iv), $|(\mathbf{1}_Y \circ f)(x)| \ovee |(\mathbf{1}_Y \circ g)(x)| \le |x|$ for all $x \in X$. Hence $|f(x)| \ovee |g(x)| \le |x|$ for all $x \in X$. By the same characterization again, we have $f \perp g$.

(viii) Let $p \in \sigma\mathbf{WMod}_M(X,M)$. Define $p^\perp\colon X \to M$ by $p^\perp(x) = |x| \ominus p(x)$, where $|x| \ominus p(x)$ is the unique element in M satisfying $(|x| \ominus p(x)) \ovee p(x) = |x|$. It is straightforward to check that $p^\perp$ is a morphism in $\sigma\mathbf{WMod}_M$, and a unique one that satisfies the required condition. □

The following lemma is the countable version of [9, Lemma 4.8] (or [10, Lemma 3.2.5]). It can be proved in the same manner as the finite case.

Lemma 63. *Let* $\mathbf{C}$ *be a* σ*-effectus, and* $\coprod_{\lambda\in\Lambda} B_\lambda$ *a countable coproduct in* $\mathbf{C}$*. There is a bijective correspondence between morphisms* $f\colon A \to \coprod_{\lambda\in\Lambda} B_\lambda$ *and families of morphisms* $(f_\lambda\colon A \to B_\lambda)_{\lambda\in\Lambda}$ *such that* $(\mathbf{1}\circ f_\lambda)_{\lambda\in\Lambda}$ *is summable in* $\mathrm{Pred}(A) = \mathbf{C}(A,I)$*. They are related via* $f_\lambda = \rhd_\lambda \circ f$*.* □

Proposition 64. *Let* $\mathbf{C}$ *be an* σ*-effectus with scalars* $M = \mathbf{C}(I,I)$*. Then the assignment* $A \mapsto \mathrm{St}_\le(A)$ *induces a morphism of* σ*-effectuses* $\mathrm{St}_\le\colon \mathbf{C} \to \sigma\mathbf{WMod}_{M^{\mathrm{op}}}$*.*

Proof. It is easy to see that the functor $\mathrm{St}_\le\colon \mathbf{C} \to \sigma\mathbf{WMod}_{M^{\mathrm{op}}}$ is well-defined. It preserves the unit object as $\mathrm{St}_\le(I) = \mathbf{C}(I,I) = M$ and the truth maps as $\mathrm{St}_\le(\mathbf{1}_X) = \mathbf{1}_X \circ (-) = |-| = \mathbf{1}_{\mathrm{St}_\le(X)}$. Lastly, it also preserves countable coproducts: we have a bijection between the underlying sets

$$\begin{aligned}
\mathrm{St}_\le\big(\textstyle\coprod_\lambda X_\lambda\big) \coloneqq \mathbf{C}\big(I,\textstyle\coprod_\lambda X_\lambda\big) &\overset{\text{Lem. 63}}{\cong} \{(\omega_\lambda)_\lambda \in \textstyle\prod_\lambda \mathbf{C}(I,X_\lambda) \mid (\mathbf{1}\circ\omega_\lambda)_\lambda \text{ is summable in } \mathbf{C}(I,I)\} \\
&= \{(\omega_\lambda)_\lambda \in \textstyle\prod_\lambda \mathrm{St}_\le(X_\lambda) \mid (|\omega_\lambda|)_\lambda \text{ is summable in } M\} \\
&=: \textstyle\coprod_\lambda \mathrm{St}_\le(X_\lambda)
\end{aligned}$$

The bijection is indeed an isomorphism in $\sigma\mathbf{WMod}_{M^{\mathrm{op}}}$. □

B Proofs in Section 4

Proof of Proposition 36. The 'only if' direction is obvious. For the 'if' direction, suppose that the effectus is substate-separated. Let $f,g\colon A \to B$ be morphisms such that $f\circ\omega = g\circ\omega$ for any $\omega \in \mathrm{St}(A)$. We need to show that then $f = g$. By substate separation it suffices to show that $f\circ\rho = g\circ\rho$ for all *substates* $\rho \in \mathrm{St}_\le(A)$. Hence, let $\rho \in \mathrm{St}_\le(A)$ be an arbitrary substate. If $\rho = 0$, then $f\circ\rho = 0 = g\circ\rho$. Otherwise, if $\rho \ne 0$, let $\overline{\rho}$ be the normalization of ρ, i.e. the state satisfying $\overline{\rho}\circ\mathbf{1}\circ\rho = \rho$. By assumption on f and g we have $f\circ\overline{\rho} = g\circ\overline{\rho}$ and hence $f\circ\rho = f\circ\overline{\rho}\circ\mathbf{1}\circ\rho = g\circ\overline{\rho}\circ\mathbf{1}\circ\rho = g\circ\rho$ as desired. □

Proof of Theorem 37. (i) $\Longrightarrow$ (ii): Already holds for regular effectuses; see [9, Proposition 6.4] or [10, Proposition 4.5.2].

(ii) $\Longrightarrow$ (iii): Suppose that $s\cdot t = 0$ and $t \ne 0$. As $s\cdot t \le t$ there is a unique $(s\cdot t)/t$ satisfying $((s\cdot t)/t)\cdot t = s\cdot t = 0$. But as both 0 and s have this property we conclude that $s = (s\cdot t)/t = 0$.

(iii) $\Longrightarrow$ (i): Let $\omega\colon I \to A$ be a nonzero substate. We write $s \coloneqq (\mathbf{1}\omega)^\perp$ and define

$$\tilde{\omega} \coloneqq \bigovee_{n=0}^{\infty} \omega\circ s^n \quad : I \longrightarrow A\,.$$

The sum is the *iteration* of the map $\kappa_1\circ s \ovee \kappa_2\circ\omega\colon I \to I + A$ and hence exists, see [32, Theorem 3.2.24].

We prove that $\tilde{\omega}$ is the normalization of ω. First, we show that $\tilde{\omega}$ is a state, i.e. a total map. Let $t \coloneqq \mathbf{1}\circ\tilde{\omega} = \bigovee_{n=0}^{\infty} s^\perp\cdot s^n$. Then

$$t \;=\; \bigovee_{n=0}^{\infty} s^\perp\cdot s^n \;=\; s^\perp \ovee \left(\bigovee_{n=0}^{\infty} s^\perp\cdot s^n\right)\cdot s \;=\; s^\perp \ovee t\cdot s\,.$$

Since $t = t\cdot(s\ovee s^\perp) = t\cdot s \ovee t\cdot s^\perp$, we obtain $t\cdot s^\perp = s^\perp$ by cancellation. Then $s^\perp = (t\ovee t^\perp)\cdot s^\perp = s^\perp \ovee (t^\perp\cdot s^\perp)$, so that $t^\perp\cdot s^\perp = 0$. Because $s^\perp = \mathbf{1}\omega \ne 0$ and there are no nontrivial zero divisors, $t^\perp = 0$, that is, $\mathbf{1}\circ\tilde{\omega} = t = 1$. Next, we have

$$\tilde{\omega}\cdot\mathbf{1}\omega \;=\; \bigovee_{n=0}^{\infty}\omega\cdot s^n\cdot s^{\perp} \;=\; \omega\cdot\bigovee_{n=0}^{\infty}s^{\perp}\cdot s^n \;=\; \omega\cdot 1 \;=\; \omega\,.$$

Here note that s and $s^{\perp}$ commute. To see the uniqueness of the normalization, let ρ be a state with $\omega = \rho\cdot\mathbf{1}\omega$ $(=\rho\cdot s^{\perp})$. Then

$$\rho \;=\; \rho\cdot 1 \;=\; \rho\cdot\Big(\bigovee_{n=0}^{\infty}s^{\perp}\cdot s^n\Big) \;=\; \bigovee_{n=0}^{\infty}\rho\cdot s^{\perp}\cdot s^n \;=\; \bigovee_{n=0}^{\infty}\omega\cdot s^n \;=\; \tilde{\omega}\,.$$

Therefore $\tilde{\omega}$ is the normalization of ω.

(iv) $\Longrightarrow$ (iii): Let $s\cdot t = 0$ for $s,t\in\mathbf{C}(I,I)$. Assume $t\neq 0$. Because t is an epi and $s\circ t = 0 = 0\circ t$, we obtain $s = 0$. This proves (iii).

(i) $\Longrightarrow$ (iv): By what we have already proved, we may assume that (ii) and (iii) hold. Let $s\colon I\to I$ be a nonzero scalar. Suppose that $\omega_1\circ s = \omega_2\circ s$ for $\omega_1,\omega_2\colon I\to A$. If $\omega_1 = 0$, then $\mathbf{1}\circ\omega_2\circ s = 0$. Since s is nonzero, we obtain $\mathbf{1}\circ\omega_2 = 0$ by (iii), and hence $\omega_2 = 0$. Similarly $\omega_2 = 0$ implies $\omega_1 = 0$. Therefore it suffices to consider the case where both ω_1 and ω_2 are nonzero. Let

$$t \;\coloneqq\; \mathbf{1}\circ\omega_1\circ s \;=\; \mathbf{1}\circ\omega_2\circ s\,.$$

By (iii) it follows that t is nonzero. By division, we have $\mathbf{1}\omega_1 = t/s = \mathbf{1}\omega_2$. By normalization, there are states $\bar{\omega}_1,\bar{\omega}_2\colon I\to X$ such that $\omega_1 = \bar{\omega}_1\circ\mathbf{1}\omega_1$ and $\omega_2 = \bar{\omega}_2\circ\mathbf{1}\omega_2$. Then

$$\begin{aligned}\bar{\omega}_1\circ t \;&=\; \bar{\omega}_1\circ\mathbf{1}\omega_1\circ s\\ &=\; \omega_1\circ s\\ &=\; \omega_2\circ s\\ &=\; \bar{\omega}_2\circ\mathbf{1}\omega_2\circ s\\ &=\; \bar{\omega}_2\circ t\,.\end{aligned}$$

Since $\bar{\omega}_1\circ t = \bar{\omega}_2\circ t$ is nonzero, $\bar{\omega}_1 = \bar{\omega}_2$ by the uniqueness of normalization. Therefore $\omega_1 = \bar{\omega}_1\circ\mathbf{1}\omega_1 = \bar{\omega}_2\circ\mathbf{1}\omega_2 = \omega_2$. □

C Proofs in Section 5

To prove Proposition 52, first we establish the connection between monotone σ-complete ordered vector spaces with order unit and ω-complete effect modules.

Lemma 65. *Let E be an ω-complete effect $[0,1]$-module. For each ascending sequence $(a_n)_{n\in\mathbb{N}}$ in E and $N\in\mathbb{N}$, we have $\bigvee_n 2^{-N}\cdot a_n = 2^{-N}\cdot\bigvee_n a_n$.*

Proof. It suffices to prove $\bigvee_n (1/2)\cdot a_n = (1/2)\cdot\bigvee_n a_n$, which implies the claim by induction. To simplify notation, we write $h = 1/2$. Let $b_n = h\cdot a_n$. As $\bigvee_n b_n \le h\cdot 1$, the sum $(\bigvee_n b_n)\ovee(\bigvee_n b_n)$ is defined. We claim that $(\bigvee_n b_n)\ovee(\bigvee_n b_n) = \bigvee_n a_n$. Indeed, $a_n = b_n\ovee b_n \le (\bigvee_n b_n)\ovee(\bigvee_n b_n)$. If $a_n\le c$, then $b_n = h\cdot a_n\le h\cdot c$ and hence $\bigvee_n b_n\le h\cdot c$. Thus $c = h\cdot c\ovee h\cdot c\ge(\bigvee_n b_n)\ovee(\bigvee_n b_n)$. Therefore

$$h\cdot\bigvee_n a_n \;=\; h\cdot\Big(\Big(\bigvee_n b_n\Big)\ovee\Big(\bigvee_n b_n\Big)\Big) \;=\; h\cdot\Big(\bigvee_n b_n\Big)\ovee h\cdot\Big(\bigvee_n b_n\Big) \;=\; \bigvee_n b_n \;=\; \bigvee_n h\cdot a_n\,.$$

□

Lemma 66. *An ordered vector space A with order unit u is monotone σ-complete if and only if the unit interval $[0,u]_A$ is ω-complete.*

Proof. The 'only if' direction is straightforward. Conversely, suppose that $[0,u]_A$ is ω-complete. Let $(a_n)_n$ be an ascending sequence in A bounded above. Let $a'_n = a_n - a_0$, so that $(a'_n)_n$ is a positive ascending sequence bounded above. We can find $N \in \mathbb{N}$ such that $(a'_n)_n$ is bounded by $2^N u$. Then $(2^{-N} \cdot a'_n)_n$ is an ascending sequence in $[0,u]_A$, so there is a supremum $\bigvee_n 2^{-N} \cdot a'_n$ in $[0,u]_A$. We will show that $2^N \cdot \bigvee_n 2^{-N} \cdot a'_n$ is a supremum of $(a'_n)_n$ in A. Clearly $a'_n \leq 2^N \cdot \bigvee_n 2^{-N} \cdot a'_n$ for each $n \in \mathbb{N}$. Suppose that $a'_n \leq b$ for each $n \in \mathbb{N}$. Then we can find $M \in \mathbb{N}$ such that $b \leq 2^M u$ and $N \leq M$. Then we have $\bigvee_n 2^{-M} \cdot a'_n \leq 2^{-M} \cdot b$, and hence

$$b \;\geq\; 2^M \cdot \bigvee_n 2^{-M} \cdot a'_n \;=\; 2^M \cdot \bigvee_n 2^{-(M-N)} \cdot 2^{-N} \cdot a'_n \;\stackrel{\star}{=}\; 2^M \cdot 2^{-(M-N)} \cdot \bigvee_n 2^{-N} \cdot a'_n \;=\; 2^N \cdot \bigvee_n 2^{-N} \cdot a'_n .$$

Here all $\bigvee$ denote suprema in $[0,u]_A$, and the equality $\stackrel{\star}{=}$ holds by Lemma 65. Therefore $(a'_n)_n$ has a supremum in A. It follows that $(a_n)_n = (a_0 + a'_n)_n$ has a supremum in A too. □

The following equivalence for morphisms can be proved similarly by translation and scaling.

Lemma 67. *Let $f\colon A \to B$ be a subunital positive linear map between monotone σ-complete ordered vector spaces with order unit. Then f is σ-normal if and only if the restriction $f\colon [0,u]_A \to [0,u]_B$ is ω-continuous.* □

In order to prove Proposition 52 we will need the following lemmas.

Lemma 68 ([40, Lemmas 1.1 and 1.2])**.** *Every monotone σ-complete ordered vector space with order unit is a Banach order-unit space.* □

Lemma 69. *Every ω-complete effect $[0,1]$-module is a σ-effect $[0,1]$-module.*

Proof. Let E be an ω-complete effect $[0,1]$-module. We need to prove that the $[0,1]$-action $\cdot\colon [0,1] \times E \to E$ is σ-biadditive. By Lemma 17, it suffices to prove ω-continuity in each argument. By Proposition 49 and Lemmas 66 and 68, we may assume that $E = [0,u]_A$ for some monotone σ-complete Banach order-unit space (A,u).

ω-continuity in the first argument: Fix $a \in [0,u]_A$. We will prove that $(-) \cdot a\colon [0,1] \to [0,u]_A$ is ω-continuous. Let $(r_n)_{n\in\mathbb{N}}$ be an ascending sequence in $[0,1]$. Clearly $(\bigvee_n r_n) \cdot a$ is an upper bound of $r_n \cdot a$. Let $b \in [0,u]_A$ satisfy $r_n \cdot a \leq b$ for all $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ be an arbitrary nonzero number. Then there is some $m \in \mathbb{N}$ such that $\bigvee_n r_n < r_m + \frac{1}{N}$, so that

$$\Big(\bigvee\nolimits_n r_n\Big) \cdot a \;\leq\; (r_m + \frac{1}{N}) \cdot a \;=\; r_m \cdot a + \frac{a}{N} \;\leq\; b + \frac{u}{N} .$$

Thus $N \cdot ((\bigvee_n r_n) \cdot a - b) \leq u$. Because N is arbitrary and A is Archimedean, we obtain $(\bigvee_n r_n) \cdot a - b \leq 0$, that is, $(\bigvee_n r_n) \cdot a \leq b$. Therefore $(\bigvee_n r_n) \cdot a = \bigvee_n (r_n \cdot a)$.

ω-continuity in the second argument: If $r = 0$, then $0 \cdot (-)\colon [0,u]_A \to [0,u]_A$ is trivially ω-continuous. Fix $r \in (0,1]$. Then $r \cdot (-)\colon A \to A$ is an order isomorphism, with the monotone inverse $r^{-1} \cdot (-)\colon A \to A$. Thus $r \cdot (-)\colon A \to A$ preserves all suprema in A, and the restriction $r \cdot (-)\colon [0,u]_A \to [0,u]_A$ is ω-continuous. □

Proof of Proposition 52. By Lemmas 66 and 67, the equivalence $\mathbf{OVSu} \simeq \mathbf{EMod}_{[0,1]}$ of Proposition 49 restricts to the category of monotone σ-complete ordered vector spaces with order unit and σ-normal subunital positive linear maps, and the category of ω-complete effect $[0,1]$-modules and ω-continuous additive maps. These two categories are respectively equal to $\sigma\mathbf{BOUS}$ and $\sigma\mathbf{EMod}_{[0,1]}$ by Lemmas 68 and 69. □

Proof of Proposition 55. The construction of the 'inverse' functor $\mathbf{CWMod}_{[0,1]} \to \mathbf{OVSt}$ is very much the same as that of $\mathbf{EMod}_{[0,1]} \to \mathbf{OVSu}$ given in [27, § 3.1]. We sketch the construction below, and refer to [10, § 7.2.1] for further details.

Let X be a cancellative weight $[0,1]$-module. The totalization [26] of the PCM X is the commutative monoid $\mathscr{T}(X) = \mathscr{M}(X)/{\sim}$ where $\mathscr{M}(X)$ is the free commutative monoid on X consisting of finite multisets on X, denoted as formal finite sums $\sum_i n_i \cdot x_i$ for $n_i \in \mathbb{N}$ and $x_i \in X$, and $\sim$ is the smallest monoid congruence such that $1 \cdot (x \ovee y) \sim 1 \cdot x + 1 \cdot y$ and $1 \cdot 0 \sim 0$. There is an embedding $X \to \mathscr{T}(X)$ given by $x \mapsto 1 \cdot x$ which is injective. Because X is a weight $[0,1]$-module, $\mathscr{T}(X)$ can be equipped with an monoid action $\mathbb{R}_{\geq 0} \times \mathscr{T}(X) \to \mathscr{T}(X)$, and the weight map extends to $|-|\colon \mathscr{T}(X) \to \mathbb{R}_{\geq 0}$. By cancellativity of X, we can prove that $\mathscr{T}(X)$ is a cancellative monoid.

We then define $V(X) = (\mathscr{T}(X) \times \mathscr{T}(X))/{\approx}$ where $\approx$ is defined by $(a,b) \approx (c,d)$ iff $a + d = b + c$. Because $\mathscr{T}(X)$ is cancellative, $\mathscr{T}(X)$ embeds into the Abelian group $V(X)$ by $a \mapsto (a,0)$. Now $V(X)$ forms a real vector space with the scalar multiplication $r(a,b) = (ra,rb)$ for $r \geq 0$ and $r(a,b) = ((-r)b,(-r)a)$ for $r < 0$. With $\mathscr{T}(X)$ embedded in $V(X)$ as a positive cone, $V(X)$ forms an ordered vector space. Moreover, $V(X)$ is positively generated and equipped with trace $\tau\colon V(X) \to \mathbb{R}$ given by $\tau(a,b) = |a| - |b|$. □

The following lemma is similar to [18, Proposition 2.4.11 and Lemma 2.4.12] and [5, Corollary 2] (see also [15]), but here stated in terms of weight modules instead of convex sets.

Lemma 70. *Let V be an ordered vector space with trace τ. Assume that the subbase $B_{\leq}(V)$ forms a σ-weight $[0,1]$-module, extending its canonical weight $[0,1]$-module structure. Then V is a Banach pre-base-norm space. Moreover, for each countable summable family $(x_n)_{n\in\mathbb{N}}$ in the σ-weight $[0,1]$-module $B_{\leq}(V)$, the series $\sum_{n=0}^{\infty} x_n$ converges to $\bigovee_{n\in\mathbb{N}} x_n$ with respect to the base norm.*

Proof. We first prove that V is a pre-base-norm space (i.e. that the seminorm is actually a norm). Let $a \in V$ satisfy $\|a\| = 0$. Let $\tilde{x},\tilde{y} \in V_+$ be such that $a = \tilde{x} - \tilde{y}$. Let $r = \max(\tau(\tilde{x}),\tau(\tilde{y}))$. If $r = 0$, we have $a = 0$. Otherwise, writing $x = r^{-1}\tilde{x}$ and $y = r^{-1}\tilde{y}$, we have $x,y \in B_{\leq}(V)$ and $r\|x - y\| = \|a\| = 0$, so that $\|x - y\| = 0$. It suffices to prove that $x - y = 0$.

By $\|x - y\| = 0$, for each $n \in \mathbb{N}$ we can find $z_n, w_n \in V_+$ such that $x - y = w_n - z_n$ and $\tau(z_n) + \tau(w_n) \leq 1/2^{n+1}$. Note that $z_n, w_n \in B_{\leq}(V)$ and by $w_n - z_n = x - y = w_{n+1} - z_{n+1}$, we have $z_n + w_{n+1} = z_{n+1} + w_n$. Because $\sum_{n\in\mathbb{N}}|z_n| + \sum_{n\in\mathbb{N}}|w_n| \leq 1$, the following countable sums exist in the σ-weight $[0,1]$-module $B_{\leq}(V)$, and the equations hold by partition-associativity.

$$\begin{aligned} z_0 \ovee \Big(\bigovee_{n=1}^{\infty} z_n\Big) \ovee \Big(\bigovee_{n=1}^{\infty} w_n\Big) &= \bigovee_{n=0}^{\infty}(z_n \ovee w_{n+1}) \\ &= \bigovee_{n=0}^{\infty}(z_{n+1} \ovee w_n) = w_0 \ovee \Big(\bigovee_{n=1}^{\infty} z_n\Big) \ovee \Big(\bigovee_{n=1}^{\infty} w_n\Big) \end{aligned}$$

By cancellation, $z_0 = w_0$, so that $x - y = w_0 - z_0 = 0$.

Before proving that V is a Banach space, we prove the claim about convergence. Let $(x_n)_n$ be a countable family summable in $B_{\leq}(V)$. Using the fact that $|x| \equiv \tau(x) = \|x\|$ for $x \in B_{\leq}(V)$ — see [18, Corollary 2.2.5] — we have for each $N \in \mathbb{N}$

$$\Big\|\Big(\bigovee_{n\in\mathbb{N}} x_n\Big) - \Big(\sum_{n=0}^{N} x_n\Big)\Big\| \;=\; \Big\|\bigovee_{n=N+1}^{\infty} x_n\Big\| \;=\; \Big|\bigovee_{n=N+1}^{\infty} x_n\Big| \;=\; \sum_{n=N+1}^{\infty} |x_n|\,.$$

Because $\lim_{N\to\infty}\sum_{n=0}^{N}|x_n| = \sum_{n=0}^{\infty}|x_n|$ and $\sum_{n=0}^{N}|x_n| + \sum_{n=N+1}^{\infty}|x_n| = \sum_{n=0}^{\infty}|x_n| < \infty$ we must have $\lim_{N\to\infty}\sum_{n=N+1}^{\infty}|x_n| = 0$. Therefore the series $\sum_{n=0}^{\infty} x_n$ converges to $\bigovee_{n\in\mathbb{N}} x_n$.

Finally we prove that V is a Banach space. It suffices to prove that every absolutely convergent series converges. Let $(x_n)_{n\in\mathbb{N}}$ be an absolutely convergent series. Without loss of generality we may assume that $\sum_{n=0}^{\infty}\|x_n\| \leq 1/2$ and $\|x_n\| \neq 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we can find $y_n, z_n \in V_+$ such that $\tau(y_n) + \tau(z_n) < 2\|x_n\|$ and $x_n = y_n - z_n$. Because $\tau(y_n) + \tau(z_n) < 2\|x_n\| \leq 1$, we have $y_n, z_n \in B_{\leq}(V)$. Moreover we have

$$\sum_{n=0}^{\infty}|y_n| \;=\; \sum_{n=0}^{\infty}\tau(y_n) \;\leq\; \sum_{n=0}^{\infty} 2\|x_n\| \;\leq\; 1$$

and similarly $\sum_{n=0}^{\infty}|z_n| \leq 1$, that is, $(y_n)_n$ and $(z_n)_n$ are summable in $B_{\leq}(V)$. Let $a = \bigovee_n y_n$ and $b = \bigovee_n z_n$. By what we have shown above, $\sum_{n=0}^{N} y_n \to a$ and $\sum_{n=0}^{N} z_n \to b$ when $N \to \infty$. Therefore $\sum_{n=0}^{N} x_n = (\sum_{n=0}^{N} y_n) - (\sum_{n=0}^{N} z_n) \to a - b$ when $N \to \infty$. □

Proof of Proposition 56. It is easy to see that for each $V \in \sigma\mathbf{BBNS}$, the subbase $B_{\leq}(V)$ forms a σ-weight $[0,1]$-module whose countable addition is given by sums of series. By this fact and Lemma 70, the equivalence $\mathbf{OVSt} \simeq \mathbf{CWMod}_{[0,1]}$ can be restricted to $\sigma\mathbf{BBNS}$ and the full subcategory of $\mathbf{CWMod}_{[0,1]}$ consisting of cancellative weight $[0,1]$-modules that have an extension to a σ-weight $[0,1]$-module. Let $\mathbf{CWMod}'_{[0,1]}$ denote this subcategory. There is a bijection between objects of $\mathbf{CWMod}'_{[0,1]}$ and $\sigma\mathbf{CWMod}_{[0,1]}$, because an extension of a weight $[0,1]$-module to a σ-weight $[0,1]$-module is unique by Lemma 70. Let $f\colon X \to Y$ be a morphism in $\mathbf{CWMod}'_{[0,1]}$. Then we can represent X and Y respectively as $B_{\leq}(V_X)$ and $B_{\leq}(V_Y)$ for some $V_X, V_Y \in \sigma\mathbf{BBNS}$, and f extends to a morphism $V_X \to V_Y$ in $\sigma\mathbf{BBNS}$. Because the countable sums in $B_{\leq}(V_X), B_{\leq}(V_Y)$ are given by convergent series and f is continuous, f preserves countable sums, i.e. it is a morphism in $\sigma\mathbf{CWMod}_{[0,1]}$. We conclude that $\mathbf{CWMod}'_{[0,1]}$ is isomorphic to $\sigma\mathbf{CWMod}_{[0,1]}$. □